Polynomial Chaos Representation of Transmission-Line Response to Random Plane Waves

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Abstract—This paper addresses the stochastic simulation of transmission lines excited by random plane wave fields. A novel methodology is presented that is based on the polynomial expansion of the forcing terms accounting for field coupling in classical transmission-line equations. Thanks to the orthogonality of polynomials, the problem is recast as a superposition of a finite number of deterministic contributions due to expansion components. This method is general and overcomes low-frequency limitations of available literature models, although allowing a faster computation of statistical parameters with respect to standard Monte Carlo (MC) sampling. The strength of the proposed technique is validated by means of comparisons with literature results and MC simulations.

Index Terms—circuit modeling, circuit simulation, field coupling, stochastic analysis, transmission lines, uncertainty.

I. INTRODUCTION

The demand for stochastic simulation tools in EMC community is drastically growing, in order to allow designers to perform uncertainty-aware analyses. In fact, modern electronic systems increasingly involve parameters that are partially unknown and therefore need to be described as random variables with proper distributions. Signal integrity (SI) or immunity issues may significantly degrade the electrical performances of an interconnect system and their assessment is unavoidably tied to the availability of accurate model for transmission cables, possibly including the effects of random variations in the involved parameters.

In this framework, different sources of variability are usually present, like fabrication tolerances and uncertain operating conditions. Design margins accounting for these variations must be set in order to ensure the device will perform well under different conditions.

Although the attention is generally drawn to inherent variations of the structure, the source of uncertainty may be external. A relevant example is provided by a cable bundle illuminated by a random impinging field. As a result, random voltages and currents are induced at the terminations and the problem is no longer deterministic, but must be addressed in a statistical way [1].

Brute-force Monte Carlo (MC) approach is a traditional tool to obtain statistical information. Although simple and robust, it is completely blind and does not exploit possible regularities of the random quantities, thus requiring a large number of samples to converge.

Pioneering work addressing the efficient statistical analysis of the effects produced by random impinging fields onto transmission lines has been reported in [2] and [3]. Analytical models were provided for the probability density functions of currents at the terminations of lines illuminated by a random plane wave. However, these models were limited to the specific case of a scalar (i.e., with one signal conductor and one reference) nearly-matched lossless line above a ground plane. Moreover, no analytical model has been derived for the case in which the angles parameterizing the direction of incidence, i.e., elevation and azimuth, are both random.

Although extension to the multiconductor case is possible for specific configurations [2], realistic applications cannot be assumed as limited to the low-frequency regime and usually involve bundles with complex and arbitrary geometries. Furthermore, the wave may more realistically be arriving from any possible direction in the angular space. Therefore, a more general model, also capable of accounting for arbitrary direction of incidence, would be highly desirable.

In the framework of stochastic analysis, the authors of this contribution proposed a novel approach to statistically characterize multiconductor transmission lines with uncertainties in the cross-sectional parameters [4], [5]. The method is based on the so-called polynomial chaos (PC) technique [6], i.e., on the expansion of the random electrical variables in terms of orthogonal polynomial functions. The original formulation was in frequency domain, while extension to time domain was subsequently achieved by means of harmonic superposition [7] or by applying the methodology directly to time-domain SPICE-like formulations [8].

Recently, the aforementioned PC approach has been suitably modified to deal with interconnects exposed to a random field excitation [9]. The methodology therein presented applies to lossy multiconductor interconnects having arbitrary geometries and has no low-frequency assumptions. Thanks to polynomial expansion of transmission-line forcing terms, the stochastic problem is rewritten in terms of a superposition of deterministic analyses carried out for each component into which the excitation is expanded.
In this paper, the strength of the proposed approach is validated by means of comparisons with literature results available in [3] and by demonstrating how their limitations concerning frequency and field parameters are overcome. The paper is organized as follows: section II briefly outlines the analytical models provided in [3]; section III summarizes the extension of PC to the case of random field incidence; finally, in section IV these two techniques are compared and additional results, only achievable with PC, are illustrated.

II. ANALYTICAL MODEL FOR SCALAR LINES

This section briefly recalls the analytical model presented in [3] for the response of a scalar transmission line driven by a random plane wave field. This model will provide a reference for the validation of the advocated PC technique.

Fig. 1. Single-conductor line above a ground plane illuminated by an uniform plane wave.

The discussion is based on the lossless line shown in Fig. 1, which is excited by an uniform plane wave. The model is founded on the low-frequency approximation of the current induced at the line terminations. Specifically, under this assumption, the magnitude of the current at the far-end current is the characteristic impedance of the line, and

\[ |I_L| = \omega PE_0 \left[ \frac{Z_C}{R_S} \right] \cos \theta \cos \psi \cos \eta + \sin \psi \sin \eta + \sin \theta \sin \eta \right], \tag{1} \]

where \( \omega \) is the angular frequency, \( E_0 \) is the field amplitude, \( Z_C \) is the characteristic impedance of the line, and

\[ P = \frac{2hL}{c_0} \frac{R_S}{(R_S + R_L)Z_C}, \tag{2} \]

\( c_0 \) being the speed of light in vacuum. The frequency limit for the validity of (1) depends on the \( Z_C/R_S \) ratio and turns out to be maximum when the line is matched, i.e., the ratio is 1.

After applying to (1) the rules for random variable transformations [10], the following expression for the probability density function (PDF) of \( |I_L| \) is obtained:

\[ f_{|I_L|}(y) = \frac{1}{\omega P} \int_{w_1(u)}^{w_2(u)} \frac{1}{w} f_{E_0}(w) f_{C} \left( \frac{y}{\omega Pw} \right) dw, \tag{3} \]

where \( f_{E_0} \) is the PDF of the field amplitude, \( f_{C} \) is the PDF of the component of (1) accounting for polarization and direction of incidence, while \( w_1 \) and \( w_2 \) are the integration limits, depending on the domains of \( f_{E_0} \) and \( f_{C} \). Again, application of random variable transformation properties for specific configurations and distributions of the field parameters allows to write closed-form expressions for (3). Readers are referred to [3] and references therein for the complete derivation of these results.

Despite the advantage of having a closed-form solution, the model provided by (3) is limited to a scalar line in the low-frequency regime and requires customized derivations. Moreover, no analytical solution for the case in which both the elevation \( \theta \) and the azimuth \( \psi \) are random is given in [3]. In the next section, a novel methodology is introduced to overcome these limitations.

III. POLYNOMIAL CHAOS METHOD OVERVIEW

This section summarizes the application of PC method to the analysis of randomly-illuminated interconnects. For the sake of comparison, the discussion is based again on the structure in Fig. 1. However, a more general and comprehensive formulation for lossy multiconductor lines is available in [9].

First, a vector \( \xi = [\ldots, \xi_i, \ldots] \) is defined that collects independent normalized random variables parameterizing any possible variation of the electric field. For instance, if the azimuth \( \psi \) is uniformly distributed in the range \([0, 2\pi]\), it can be expressed as

\[ \psi = \pi + \pi \xi_i, \tag{4} \]

where \( \xi_i \) is a normalized uniform random variable in the interval \([-1, 1]\).

The external field couples onto the transmission line, thus inducing voltages and currents that will depend on \( \xi \). This yields a stochastic version of the classical telegrapher’s equations [11] describing the situation depicted in Fig. 1:

\[ \frac{d}{dz} V(z, s, \xi) = -s LI(z, s, \xi) + V_F(z, s, \xi), \tag{5a} \]

\[ \frac{d}{dz} I(z, s, \xi) = -s CV(z, s, \xi) + I_F(z, s, \xi), \tag{5b} \]

where \( z \) is the direction of propagation, \( s \) is the Laplace variable, while \( L \) and \( C \) are the per-unit-length (p.u.l) inductance and capacitance of the wire, respectively. In this case, the line is assumed to be deterministic, i.e., the geometry is fixed and therefore no random variable is associated to the p.u.l. parameters. Forcing functions \( V_F \) and \( I_F \) involve integrations of the incident field and, this being random, they are random themselves and therefore dependent on \( \xi \). From now on, the dependence on the Laplace variable will be omitted for notational convenience.
As an alternative to MC sampling of (5), according to PC theory, the random voltage and current variables can be expanded with respect to \( \xi \) using a truncated series of orthogonal basis functions [12]. For instance, the one for the unknown current writes

\[
I(z, \xi) = \sum_{k=0}^{P} I_k(z) \phi_k(\xi),
\]

where \( \{\phi_k(\xi)\} \) represents a class of suitable multivariate orthogonal polynomials, depending on the distribution assumed for the random variables. Hermite and Legendre polynomials turn out to be the optimal choices when \( \xi \) are Gaussian or uniform, respectively [6]. Each class of polynomials carries a proper definition of the inner product, according to which

\[
< \phi_j, \phi_k >= \alpha_k \delta_{kj}.
\]

The above equation suggests that these polynomials are generally orthogonal but not orthonormal. In presence of multiple random variables, to preserve orthogonality, multivariate polynomials are built as a product combination of univariate ones.

Substitution of (6) and analogous expansions for the remaining variables into (5) yields:

\[
\frac{d}{dz} \sum_{k=0}^{P} V_k(z) \phi_k(\xi) = -sL \sum_{k=0}^{P} I_k(z) \phi_k(\xi) + \sum_{k=0}^{P} V_{F,k}(z) \phi_k(\xi)
\]

\[
\frac{d}{dz} \sum_{k=0}^{P} I_k(z) \phi_k(\xi) = -sC \sum_{k=0}^{P} V_k(z) \phi_k(\xi) + \sum_{k=0}^{P} I_{F,k}(z) \phi_k(\xi).
\]

Due to PC expansion, the random unknown voltage and current are expressed as analytical polynomial functions, whose coefficients are still unknown. On the other hand, the coefficients \( V_{F,k} \) and \( I_{F,k} \) of the forcing functions can be computed by means of standard numerical integration techniques. Gaussian quadratures turn out to be particularly convenient due to their mathematical definition.

According to (7), the analytical projection of (8) onto the polynomial basis itself allows to obtain pertinent equations for each voltage and current coefficient:

\[
\frac{d}{dz} V_k(z) = -sL I_k(z) + V_{F,k}(z)
\]

\[
\frac{d}{dz} I_k(z) = -sC V_k(z) + I_{F,k}(z),
\]

\( \forall k = 0, \ldots, P \). Each equation is deterministic and coincides with the solution of the line for the excitation by the \( k \)th component of the forcing functions. Therefore, the polynomial approximation in (6) turns out to be a superposition of the analyses carried out for each single component, in analogy to Fourier decomposition of signals into harmonics, and can be used to obtain statistical information by means of numerical techniques or random variable transformation properties.

The total number of components is given by

\[
P + 1 = \frac{(p + n)!}{p!n!},
\]

where \( p \) is the maximum order of univariate polynomials in the expansion and sets the expansion accuracy, while \( n \) is the number of random variables. Although the new number of unknowns is \( P + 1 \) times larger, the solution can be much faster than running a large number of MC simulations.

The generic solution of (9) is given by the chain-parameter matrix \( T \), which relates the voltages and currents at line terminations as follows:

\[
V_k(\mathcal{L}) = T_{11}(\mathcal{L}) V_0(0) + T_{12}(\mathcal{L}) I_0(0) + V_{F,T,k}(\mathcal{L})
\]

\[
I_k(\mathcal{L}) = T_{21}(\mathcal{L}) V_0(0) + T_{22}(\mathcal{L}) I_0(0) + I_{F,T,k}(\mathcal{L}),
\]

where \( \mathcal{L} \) is the interconnect length and

\[
T(\mathcal{L}) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \expm \left(-s \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix} \mathcal{L} \right).
\]

\( V_{F,T,k} \) and \( I_{F,T,k} \) are convolution terms between the forcing functions and the chain-parameter matrix [11]:

\[
V_{F,T,k}(\mathcal{L}) = \int_0^\mathcal{L} T_{11}((\mathcal{L} - \tau) V_{F,k}(\tau) + T_{12}((\mathcal{L} - \tau) I_{F,k}(\tau)) d\tau
\]

\[
I_{F,T,k}(\mathcal{L}) = \int_0^\mathcal{L} T_{21}((\mathcal{L} - \tau) V_{F,k}(\tau) + T_{22}((\mathcal{L} - \tau) I_{F,k}(\tau)) d\tau.
\]

Usually, the intermediate computation of \( V_{F,k} \) and \( I_{F,k} \) is not strictly necessary, since \( V_{F,T,k} \) and \( I_{F,T,k} \) can be directly related to the incident electric field. For instance, in the case of uniform plane wave incidence, they can be readily calculated by numerically integrating the following analytical expressions:

\[
V_{F,T}(\mathcal{L}, \xi) = \frac{1}{2j\omega C} M^+(\xi) - E_T(\mathcal{L}, \xi) + \frac{1}{2j\omega C} N^+(\xi)
\]

\[
I_{F,T}(\mathcal{L}, \xi) = -\frac{1}{2\gamma} M^-(\xi) - \frac{1}{2\gamma} N^-(\xi),
\]

where \( \gamma = j\omega \sqrt{LC} \) and \( E_T \) is the transverse field contribution. \( M^\pm \) and \( N^\pm \) are also functions of the incident field. It should be noted that in (14) the dependence on \( \xi \) denotes once again the parameters affected by variability.

According to the angle definitions, the general expression for the electric field writes

\[
\vec{E} = E_0(\epsilon_x \vec{a}_x + \epsilon_y \vec{a}_y + \epsilon_z \vec{a}_z)e^{-j\beta_x z}e^{-j\beta_y \theta}e^{-j\beta_z \psi},
\]

where

\[
\begin{align*}
\epsilon_x &= \sin \eta \sin \theta \\
\epsilon_y &= -\sin \eta \cos \theta \cos \psi - \cos \eta \sin \psi \\
\epsilon_z &= -\sin \eta \cos \theta \sin \psi + \cos \eta \cos \psi
\end{align*}
\]

and

\[
\begin{align*}
\beta_x &= -\beta \cos \theta \\
\beta_y &= -\beta \sin \theta \cos \psi \\
\beta_z &= -\beta \sin \theta \sin \psi,
\end{align*}
\]
\[ \beta = \omega / c_0 \] being the phase constant for the incident wave. Although omitted for notational convenience, it is evident that all the parameters introduced above depend themselves on \( \xi \).

The field-dependent parameters can be expressed as

\[ E_T(\mathcal{L}) = 2E_0 c_z h \left( \sin \beta_z h \ e^{-j \beta_z h} \right) e^{-j \beta_z \mathcal{L}}, \]  
(18)

\[ M^\pm = -2j E_0 c_z \left[ e^{\gamma \mathcal{L} L} \left( \frac{1}{\gamma} - \frac{1}{\gamma - j \beta_z} \right) e^{-j \beta_z h} \right. \]  
\[ \left. \times e^{-\gamma \mathcal{L} (\gamma - j \beta_z) h} \right] \times j \omega C \sin(\beta_z h) \]  
(19)

and

\[ N^\pm = 2E_0 \left( e^{\gamma \mathcal{L}} \pm e^{-\gamma \mathcal{L}} \right) e_z h \sin \beta_z h \ e^{-j \beta_z h}. \]  
(20)

Readers are referred to [11] (ch. 7) for a thorough description of the derivation of these formulas.

Finally, the sought-for voltage and current coefficients can be computed by imposing boundary conditions to (11). For instance, if line terminations are represented by means of Thévenin equivalents:

\[ V_k(0) = -R_S I_k(0) \]  
(21a)

\[ V_k(\mathcal{L}) = R_L I_k(\mathcal{L}), \]  
(21b)

where no lumped sources have been considered, the solution of (11) is given by

\[ I_k(0) = \frac{V_{FT,k}(\mathcal{L}) - R_L I_{FT,k}(\mathcal{L})}{T_{11} R_S + R_L T_{22} - T_{12} - R_L T_{21} R_S} \]  
(22a)

\[ I_k(\mathcal{L}) = I_{FT,k}(\mathcal{L}) + (T_{22} - T_{21} R_S) I_k(0), \]  
(22b)

while voltages can be obtained from (21).

IV. Validation

In this section, the PC model is used to reproduce the results available in [3] and based on the derivation outlined in section II. Moreover, additional results referring to high-frequency coupling and completely-arbitrary direction of incidence will be provided, thus proving the improvements achieved with the advocated technique. Results obtained from MC simulations will provide an additional reference.

In this framework, the wire in Fig. 1 is 1-m long and is placed at a height of \( h = 10 \) cm. Moreover, it has a radius of \( r_w = 0.5 \) mm and is loaded at both sides with its characteristic impedance, i.e.,

\[ R_S = R_L = Z_C = \frac{1}{2\pi} \mu_0 \acosh \left( \frac{h}{r_w} \right) \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 360 \Omega. \]  
(23)

Three cases will be considered: (a) random field amplitude and elevation; (b) random field amplitude and polarization; (c) random elevation and azimuth. The latter case is particularly relevant, since it refers to a field arriving from each possible direction in the space, and because no solution for it was given in [3].

A. Random Field Amplitude and Elevation

In the first case considered, the wave has random field amplitude \( E_0 \) and elevation \( \theta \), while polarization \( \eta \) and azimuth \( \psi \) are considered to be fixed. \( E_0 \) has a beta distribution with parameters \( \nu_1 = 2 \) and \( \nu_2 = 5 \) in the interval \([0, 1] \) V/m. The elevation is also unknown, thus ranging from 0 to \( \pi/2 \) (since the line lies above a ground plane, the elevation cannot be greater than this value). The wave is assumed to be impinging from the left side (\( \psi = -\pi/2 \)) and polarized in the incidence plane (\( \eta = \pi/2 \)).

Fig. 2 shows the magnitude of the induced current at the far-end termination. The solid black line is its mean value computed by means of 20 000 MC simulations, while crosses represent the estimation provided by PC. Additionally, a limited set of MC samples is also plotted to provide a qualitative idea of the behavior of the response due to the field randomness.

Fig. 3 provides two probability density functions computed at different frequencies, i.e., 50 MHz and 800 MHz, which are also identified by the vertical dashed lines in Fig. 2. Results from both PC and MC simulations are reported, showing excellent agreement. At 50 MHz the electrical line length is still small, therefore the analytical model given by (3) can be used for the PDF computation, providing the same result as the MC simulation (top panel). Nonetheless, the PC approach extends the application to higher frequencies maintaining good accuracy (bottom panel).

B. Random Field Amplitude and Polarization

In the second case, random polarization is considered besides field amplitude. The direction of incidence of the propagating wave is supposed to be known and corresponding to \( \theta = \pi/6 \) and \( \psi = \pi/4 \). The field amplitude \( E_0 \) has the same beta distribution as the previous case, while the polarization angle \( \eta \) is uniform in the range \([-\pi, \pi]\). For this situation, closed-form solution for (3) is available as well.
Fig. 3. Probability density function of the magnitude of the far-end induced current, computed at two different frequencies for case (a). The distributions marked MC refer to 20,000 MC simulations, while those marked PC refer to the response obtained by means of PC expansion. In the top panel, MC simulation is consistent with literature results [3].

Fig. 4. Magnitude of the far-end induced current for case (b). Same comments of Fig. 2 apply here.

Fig. 5. Probability density function of the magnitude of the far-end induced current, computed at two different frequencies for case (b). Same comments of Fig. 3 apply here.

Fig. 6. Magnitude of the far-end induced current for case (c). Same comments of Figs. 2 and 4 apply here.

C. Random Elevation and Azimuth

The two examples reported in the previous sections clearly show how PC overcomes the low-frequency limitations of the models in [3]. In addition, with the proposed PC approach, it is possible to account for the case in which the direction of incidence is totally arbitrary, that of course represents a very realistic situation.

Therefore, as a further validation of the proposed approach, the following case is considered: elevation and azimuth are random in the range $[0, \pi/2]$ and $[-\pi, \pi]$, respectively. The field amplitude is deterministically assumed to be 1 V/m, while the polarization is set to $\eta = \pi/2$. It should be noted that the cable cross-section being symmetric, in principle nothing changes whether the field incises from the left side or from the right side. Hence, the actual range of variation of the azimuth can be actually halved.

Analogously to the previous cases, Fig. 6 reports the average value of the magnitude of the induced current as well as a reduced set of MC samples, showing that especially at high frequencies the responses are very chaotic and can be completely different from each other. In addition, the PDFs in
In all the three considered cases, PC exhibits excellent accuracy in reproducing any statistical parameter of interest. In particular, the accuracy in reproducing distribution shapes that greatly differ from the original uniform distribution is remarkable.

Finally, to address the PC efficiency in terms of computational times, Tab. I reports some figures summarizing the CPU time required for the simulation of the three examples on a standard laptop. The reported times are referred to an entire frequency sweep over 401 points. It can be observed that PC largely outperforms MC in terms of computational speed, while providing comparable accuracy. Clearly, MC simulation time is the same for all the three cases, since the number of samples and frequency points does not change. On the contrary, a different number of PC expansion terms may be required for each situation, thus making the CPU times different.

**Fig. 7** once again reveal that the accuracy of PC approach is not limited to low frequencies.

**Fig. 7.** Probability density function of the magnitude of the far-end induced current, computed at two different frequencies for case (c). The distributions marked MC refer to 20 000 MC simulations, while those marked PC refer to the response obtained by means of PC expansion.

**TABLE I**

<table>
<thead>
<tr>
<th>Method</th>
<th>Case</th>
<th>Simulation time</th>
<th>Speed-up</th>
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</thead>
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<tr>
<td>MC</td>
<td>–</td>
<td>10 min 13 s</td>
<td>–</td>
</tr>
<tr>
<td>PC</td>
<td>(a)</td>
<td>9.7 s</td>
<td>63×</td>
</tr>
<tr>
<td>PC</td>
<td>(b)</td>
<td>2.6 s</td>
<td>235×</td>
</tr>
<tr>
<td>PC</td>
<td>(c)</td>
<td>19.4 s</td>
<td>31×</td>
</tr>
</tbody>
</table>

**V. CONCLUSIONS**

This paper presents an efficient model for the stochastic simulation of a transmission line illuminated by a random impinging plane wave. The approach is based on the PC technique and therefore on the expansion of random voltage and current variables in terms of a series of orthogonal polynomials. By means of orthogonality properties, the stochastic problem is recast as a superposition of deterministic analyses carried out for each component which the forcing functions are expanded into. As such, it turns out to be similar to Fourier analysis for time-domain simulations. The superposition defines a known analytical function that can be used for a fast computation of any statistical parameter of interest.

In order to validate and demonstrate the strength of the advocated methodology, PC is used to reproduce available literature results. Not only it is shown to provide comparable accuracy with respect to analytical models and MC simulations, but it is much more efficient than the latter, allowing to achieve speed-ups in the range from 31 to 235. On the other hand the approach, being based on the expansion of the classical governing equations, turns out not to be affected by the low-frequency limitations that can be found in state-of-the-art analytical models. Furthermore, although not shown in this paper, the method applies also to arbitrary lossy and multiconductor interconnect geometries.

**REFERENCES**